New Cheeger bounds on eigenvalues of Markov chains

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(Discrete) Markov Chains

- Random walk on a graph.
- Stochastic matrix P: row sums 1.
- Irreducible $(\forall i, j \exists t : \mathsf{P}^t_{ij} > 0)$

  $\Rightarrow$ unique stationary distribution $\pi \, \mathsf{P} = \pi$.

- Time-reversible $(\pi_i \, \mathsf{P}_{ij} = \pi_j \, \mathsf{P}_{ji})$

  $\Rightarrow 1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -1$

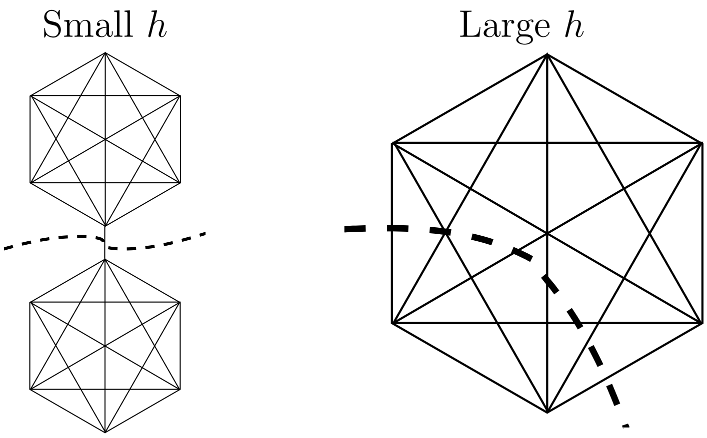
Mixing

- Irreducible + Aperiodic $\Rightarrow \mathsf{p}^{(t)} \xrightarrow{t \to \infty} \pi$
- Variation distance $\|\sigma - \pi\|_{TV} = \dfrac{1}{2} \sum_{v \in V} |\sigma(v) - \pi(v)|$.
- $\Delta(t) = \max_{\sigma_0} \|\sigma_0 \, \mathsf{P}^t - \pi\|_{TV} = \max_{x \in V} \|\mathsf{P}^t(x, \cdot) - \pi\|_{TV}$
- $\dfrac{1}{2} \lambda^t_{max} \leq \Delta(t) \leq \dfrac{1}{2} \lambda^t_{max} \sqrt{\dfrac{1 - \pi_*}{\pi_*}}$ with $\lambda_{max} = \max\{\lambda_2, \, |\lambda_n|\}$
- Math 8843 notes

Cheeger's Inequality

- (Ergodic) flow : $\mathsf{Q}(A, C) = \sum_{i \in A, j \in C} \pi_i \mathsf{P}_{ij}$.
- *Cheeger constant* :

$$h = \min_{A \subset V} \frac{\mathsf{Q}(A, A^c)}{\min\{\pi(A),\, \pi(A^c)\}}$$



Cheeger Inequality

(Lawler-Sokal '88, Jerrum-Sinclair '89)

$$1 - 2h \leq \lambda_2 \leq 1 - h^2/2$$

(Chung, Houdré-Tetali, ...)

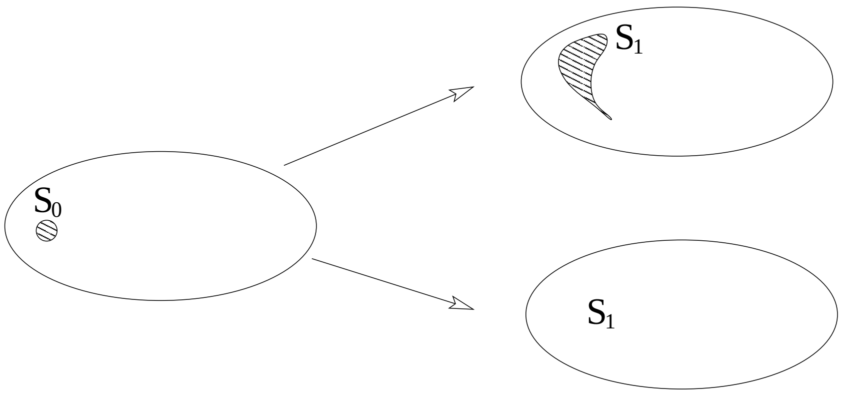
$$\lambda_2 \leq \sqrt{1 - h^2}$$

Evolving Sets

Morris and Peres (STOC 2003) “Evolving sets and mixing”

- $S_0 = \{x\}$
- Choose $u \in [0,1]$ u.a.r.
- Let $A = S_n$. Then

$$S_{n+1} = A_u = \{v \in V : \mathsf{Q}(A,v) \geq u \cdot \pi(v)\}$$



Key Properties

- $\mathsf{P}^t(x,y) = \dfrac{\pi(y)}{\pi(x)} \mathsf{P}(y \in S_t \mid S_0 = \{x\})$
- $\mathsf{E}\pi(S_t) = \pi(S_0)$
- If $\mathcal{M}$ is lazy ($\forall v \in V : \mathsf{P}(v,v) \geq 1/2$) then

$$\mathsf{Q}(A, A^c) = \int_0^{1/2} (\pi(A_u) - \pi(A))\, du = \int_{1/2}^{1} (\pi(A) - \pi(A_u))\, du$$

Mixing

Chi-Square Mixing

$$\begin{aligned}\Delta(t) &= \max_{x\in V} \|\mathsf{P}^t(x,\cdot)-\pi\|_{TV} \le \max_{x\in V} \frac{1}{2}\,\|\mathsf{P}^t(x,\cdot)-\pi\|_{\chi^2(\pi)}^{1/2} \\ &\le \max_{x\in V} \frac{1}{2\pi(x)}\,\mathsf{E}\sqrt{\min\{\pi(S_t),\,1-\pi(S_t)\}} \\ &\le \frac{\psi^t}{2\sqrt{\pi_*}} \quad \textit{with} \quad \psi = \max_{\substack{A\subset V,\\ \pi(A)\le 1/2}} \int_0^1 \sqrt{\frac{\pi(A_u)}{\pi(A)}}\,du\,.\end{aligned}$$

Cheeger Corollary

$$\lambda_2 \le \lambda_{max} \le \psi$$

If $\mathcal{M}$ is lazy

$$\psi \le \frac{\sqrt{1+2h}+\sqrt{1-2h}}{2} \le 1-\frac{h^2}{2}$$

Proof: Apply $\frac{1}{2}\,\lambda_{max}^t \le \Delta(t)$. Then use Jensen's inequality.

Montenegro "Evolving set bounds on various mixing quantities"

Mixing Theorem

$$\|\mathsf{P}^t(x,\cdot)-\pi\|_{TV} \le \frac{1}{\pi(x)}\,\mathsf{E}\pi(S_t)(1-\pi(S_t))$$

$$Ent_\pi\left(\frac{\mathsf{P}^t(x,\cdot)}{\pi}\right) \le \frac{1}{\pi(x)}\,\mathsf{E}\pi(S_t)\log\frac{1}{\pi(S_t)}$$

$$\|\mathsf{P}^t(x,\cdot)-\pi\|_{\chi^2(\pi)}^{1/2} \le \frac{1}{\pi(x)}\,\mathsf{E}\sqrt{\pi(S_t)(1-\pi(S_t))}$$

Extending Cheeger

Montenegro "Evolving set and Cheeger bounds on eigenvalues of Markov chains"

Theorem Given $f : (0,1) \to \mathbb{R}_{>0}$ let $f(0) := f(1) := 0$. Then

$$\lambda_{max} \leq \mathcal{C}_f = \max_{A \subset V} \int_0^1 \frac{f(\pi(A_u))}{f(\pi(A))}\, du$$

If $\forall x \in (0, 1/2) : f(x) \leq f(1-x)$ then consider only $\pi(A) \leq 1/2$.

Proof: Let $M = \max_{\substack{A \subset V, \\ \pi(A) \neq 0,1}} \frac{\sqrt{\min\{\pi(A),\, 1-\pi(A)\}}}{f(\pi(A))} \leq 1/\sqrt{2}$. Then

$$\begin{aligned} \|\mathsf{P}^t(x,\cdot) - \pi\|_{TV} &\leq \frac{1}{2\pi(x)}\, \mathsf{E} \sqrt{\min\{\pi(S_t),\, 1-\pi(S_t)\}} \\ &\leq \frac{M}{\pi(x)}\, \mathsf{E}\, f(\pi(S_t)) \\ &\leq \frac{M\, f(\pi(x))}{\pi(x)}\, \mathcal{C}_f^t. \end{aligned}$$

$$\frac{1}{2}\, \lambda_{max}^t \leq \Delta(t) \leq \left(\max_{x \in V} \frac{M\, f(\pi(x))}{\pi(x)} \right) \mathcal{C}_f^t$$

$$\implies \lambda_{max} \leq \mathcal{C}_f$$

Cheeger Corollary

$$\lambda_2 \leq \lambda_{max} \leq \psi$$

Proof: Let $f(x) = \sqrt{x}$. Then use Jensen's inequality.

Separating λ_2 from λ_n

- Given general (non-lazy) P. Consider $\mathsf{P}' = \dfrac{1}{2}\,(I + \mathsf{P})$.

$$\lambda_i' = \frac{1}{2}\,\lambda_i + \frac{1}{2} \qquad\qquad \Rightarrow \lambda_2 \le -1 + 2\,\mathcal{C}_f'$$

- Given P with $\forall v \in V :\, \mathsf{P}(v,v) \ge \gamma$. Consider $\mathsf{P}'' = \frac{1}{1-\gamma}\,\mathsf{P} + \left(1 - \frac{1}{1-\gamma}\right)\, I$.

$$\lambda_i'' = \frac{1}{1-\gamma}\,\lambda_i - \frac{\gamma}{1-\gamma} \qquad\qquad \Rightarrow \lambda_n \ge -(1-\gamma)\,\mathcal{C}_f'' + \gamma$$

Example: Morris-Peres bound for lazy chain is $\lambda_2 \le 1 - h^2/2$.

Consider general $\mathcal{M}$.

Rescale. Then $\mathsf{Q}'(A, A^c) = \frac{1}{2}\,\mathsf{Q}(A, A^c)$ so $h' = h/2$.

$$\Longrightarrow \lambda_2 \le -1 + 2\,\mathcal{C}_f' \le 1 - h^2/4$$

Extending Cheeger

Corollary Given $f:\,(0,1)\to\mathbb{R}_{>0}$ concave let $f(0):=f(1):=0$. Then

$$\lambda_2 \;\le\; -1+2\max_{A\subset V}\frac{f(\pi(A)+\mathsf{Q}(A,A^c))+f(\pi(A)-\mathsf{Q}(A,A^c))}{2\,f(\pi(A))}$$

$$\lambda_n \;\ge\; -\max_{A\subset V}\frac{\wp\, f\left(\pi(A)+\frac{\Psi(A)}{\wp}\right)+(1-\wp)\,f\left(\pi(A)-\frac{\Psi(A)}{1-\wp}\right)}{f(\pi(A))}$$

where $\Psi(A)=\frac{1}{2}\int_0^1|\pi(A)-\pi(A_u)|\,du$ and $\wp=\sup\{y:\,\pi(A_y)\ge\pi(A)\}$.

If $\forall x\in[0,1/2]:\,f(x)\le f(1-x)$ then it suffices to consider $\pi(A)\le 1/2$.

Proof: Apply Jensen and rescale for λ_2. Apply Jensen directly for λ_n.

Remarks

- $\mathsf{Q}(A,A^c)$ is ergodic flow from A to A^c.
  $\Psi(A)$ is smallest flow from A to a set of size $\pi(A^c)$.
- For lazy chains $\wp=1/2$ and $\Psi(A)=\mathsf{Q}(A,A^c)$.
- $\mathsf{Q}(A,A^c)=0$ iff disconnected.
  $\Psi(A)=0$ iff A is a bipartition.

Examples

- **Periodic walk on uniform two-point space**

  Corollary is always sharp ($-1\le\lambda_n=\lambda_2\le -1$).

- **Complete graph or weighted two-point space**

  $\mathcal{C}_f$ form is always sharp if $f(x)$ symmetric.

Cheeger-like Inequalities

Corollary

$$
\begin{aligned}
1-\tilde{h} \;\le\; \lambda_2 \le\; & -1+2\sqrt{1-\tilde{h}^2/4} \;\le 1-\frac{\tilde{h}^2}{4} \\
\lambda_n \ge\; & \quad -\sqrt{1-\tilde{\phi}^2} \qquad \ge -1+\frac{\tilde{\phi}^2}{2}
\end{aligned}
$$

where

$$
\tilde{h} = \min_{A\subset V} \frac{\mathsf{Q}(A,A^c)}{\pi(A)\pi(A^c)} \quad \textit{and} \quad \tilde{\phi} = \min_{A\subset V} \frac{\Psi(A)}{\pi(A)\pi(A^c)}\,.
$$

Proof: Let $f(x)=\sqrt{x(1-x)}$ and simplify.

Corollary

$$
\begin{aligned}
\lambda_2 &\le 1-\mathsf{g}_1^2 \\
\lambda_n &\ge -1+2\,\mathsf{g}_2^2
\end{aligned}
$$

where

$$
\mathsf{g}_1 = \min_{\pi(A)\le 1/2} \frac{\mathsf{Q}(A,A^c)}{\pi(A)\sqrt{1+\log(1/2\pi(A))}} \quad \textit{and} \quad \mathsf{g}_2 = \min_{\pi(A)\le 1/2} \frac{\Psi(A)}{\pi(A)\sqrt{1+\log(1/2\pi(A))}}\,.
$$

Proof: Let $f(x)=x\,(1+\log(1/2x))$ and simplify.

Corollary

$$
\begin{aligned}
\lambda_2 &\le \max_{A\subset V} \cos(2\pi\,\mathsf{Q}(A,A^c)) \\
\lambda_n &\ge -\max_{A\subset V} \cos(2\pi\,\Psi(A)) \text{ if } \wp = 1/2 \text{ for all } A\subset V
\end{aligned}
$$

Proof: Let $f(x)=\sin(\pi\,x)$ and simplify.

How to choose $f(x)$?

Lemma If f and f'' are concave then

$$\lambda_2 \le 1 - \min_{A \subset V} \frac{-f''(\pi(A))}{f(\pi(A))} \, \mathsf{Q}(A, A^c)^2 \, .$$

Proof: Check that $f(x+y) + f(x-y) \le 2f(x) + f''(x)\, y^2$ if f and f'' are concave.

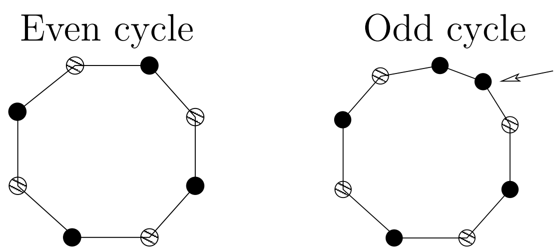
Strategy Given $\mathsf{Q}(A, A^c) \ge C\, g(\pi(A))$, let $c > 0$ and solve

$$g(x)^2\, f''(x) + c\, f(x) = 0$$

If $f : (0,1) \to \mathbb{R}_{>0}$ and f and f'' concave then

$$\lambda_2 \le 1 - c\, C^2$$

Simple walk on cycle C_n



- $\underline{n \text{ even}}$: $\mathsf{Q}(A, A^c) \ge 1/n$, $\Psi = 0$

  Solution to $f''(x) + \pi^2 f(x) = 0$ is $f(x) = \sin(\pi\, x)$.

  $$\lambda_2 \le \cos(2\pi/n) \quad \text{and} \quad \lambda_n \ge -1$$

- $\underline{n \text{ odd}}$: $\mathsf{Q}(A, A^c) \ge 1/n$, $\Psi(A) \ge 1/2n$ with $\wp = 1/2$

  $$\lambda_2 \le \cos(2\pi/n) \quad \text{and} \quad \lambda_n \ge -\cos(\pi/n)$$

All are correct values!

Vertex Expansion

Alon, Houdré-Tetali, Stoyanov, ...

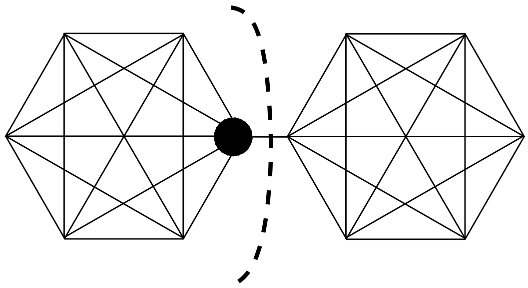
λ_2 for a graph in terms of

- $\partial_{in}(A) = \{x \in A : \mathsf{P}(x, A^c) > 0\}$, $\quad \partial_{out}(A) = \{x \in A^c : \mathsf{P}(x, A) > 0\}$
- $h_{in} = \min_{\substack{A \subset V \\ \pi(A) \leq 1/2}} \frac{\pi(\partial_{in}(A))}{\pi(A)}$, $\quad h_{out} = \min_{\substack{A \subset V \\ \pi(A) \leq 1/2}} \frac{\pi(\partial_{out}(A))}{\pi(A)}$
- $\mathsf{P}_0 = \min_{x,y \in V} \{\mathsf{P}(x,y) : \mathsf{P}(x,y) > 0\}$.

Max-degree walk on graph

$$\begin{aligned} 1 - \lambda_2 &\geq \max\left\{ \frac{\mathsf{P}_0}{2} \left(1 - \sqrt{1 - h_{in}}\right)^2, \frac{\mathsf{P}_0}{4} \left(\sqrt{1 + h_{out}} - 1\right)^2 \right\} \\ &\geq \max\left\{ \frac{\mathsf{P}_0\, h_{in}^2}{8}, \frac{\mathsf{P}_0\, h_{out}^2}{16} \left(1 - \frac{h_{out}}{4}\right)^2 \right\} \end{aligned}$$

Example Max degree walk on Barbell $K_n - K_n$.



$h = \frac{1}{n^2} \Rightarrow 1 - \lambda_2 \geq \frac{1}{2n^4}$

$h_{in} = 1/n,\ \mathsf{P}_0 = 1/n \Rightarrow 1 - \lambda_2 \geq \frac{1}{8\, n^3}$.

If $h = \mathsf{P}_0\, h_{in}$ then

$$1 - \lambda_2 \geq \frac{h^2}{8\mathsf{P}_0}$$

Much bigger than Cheeger if P_0 small.

Easy Inequality Prover

Lemma If $f : [0,1] \to \mathbb{R}$ concave, two non-increasing functions $g, g' : [0,1] \to [0,1]$ satisfy $\int_0^1 g(u)\,du = \int_0^1 g'(u)\,du$ and $\forall t \in [0,1] : \int_0^t g(u)\,du \geq \int_0^t g'(u)\,du$, then

$$\int_0^1 f \circ g(u)\,du \leq \int_0^1 f \circ g'(u)\,du\,.$$

Strategy

- Given initial conditions (e.g. $\mathcal{M}$ is lazy and $\pi(A)$ and $\mathsf{Q}(A, A^c)$ known) let $g(u) = \pi(A_u)$.
- $\exists g(u)$ satisfying conditions and minimizes $\int_0^t g(u)\,du$?
- If so then this maximizes $\mathcal{C}_f$ for any choice of $f(x)$ concave!

Example Chain $\mathcal{M}$ lazy and $\pi(A)$ and $\mathsf{Q}(A, A^c)$ known.

- Average value of $\pi(A_u)$ for $u \leq 1/2$ is $\pi(A) + \frac{\mathsf{Q}(A,A^c)}{1/2}$.
- $\pi(A_u)$ decreasing $\Rightarrow \forall t \leq 1/2 : \displaystyle\int_0^t \pi(A_u)\,du$ dominates average.
- Then $\pi(A_u) = \pi(A) + 2\mathsf{Q}(A, A^c)$ for all $u \in [0, 1/2]$ will maximize $\mathcal{C}_f$.
- Likewise, $\pi(A_u) = \pi(A) - 2\mathsf{Q}(A, A^c)$ for all $u \in (1/2, 1]$ maximizes $\mathcal{C}_f$.

$\Longrightarrow \mathcal{C}_f$ maximized by

$$\pi(A_u) = \begin{cases} \pi(A) - 2\mathsf{Q}(A, A^c) & \text{if } u > 1/2 \\ \pi(A) + 2\mathsf{Q}(A, A^c) & \text{if } u \leq 1/2 \end{cases}$$

for every concave f.

Easy Inequalities

Lazy and h known
Worst case $\pi(A_u)$ just shown. Then

$$\begin{aligned}\lambda_2 \le \mathcal{C}_{\sqrt{x}} &\le \max_{\pi(A)\le 1/2} \frac{1}{2}\sqrt{\frac{\pi(A)+2\mathsf{Q}(A,A^c)}{\pi(A)}} + \frac{1}{2}\sqrt{\frac{\pi(A)-2\mathsf{Q}(A,A^c)}{\pi(A)}} \\ &= \frac{\sqrt{1+2h}+\sqrt{1-2h}}{2}\end{aligned}$$

ϕ known

- Fix $\pi(A)$. Then $\Psi(A) = \phi(A)\,\pi(A)$.
- Consider intervals $u \in [0,\wp]$ and $u \in [\wp,1]$.

$\Longrightarrow \mathcal{C}_f$ maximized by

$$\pi(A_u) = \begin{cases} \pi(A) - \frac{\Psi(A)}{1-\wp} & \text{if } u > \wp \\ \pi(A) + \frac{\Psi(A)}{\wp} & \text{if } u \le \wp \end{cases}$$

$\Longrightarrow$ Cheeger bound on λ_n.

Given h_{in}, h_{out} and h
Assume $\mathcal{M}$ is lazy. Worst case is

$$\pi(A_u) = \begin{cases} \pi(A) - h_{in}\,\pi(A) & \text{if } u \in [1-\mathsf{P}_0, 1] \\ \pi(A) - 2\pi(A)\,\frac{h-\mathsf{P}_0 h_{in}}{1-2\mathsf{P}_0} & \text{if } u \in [1/2, 1-\mathsf{P}_0] \\ \pi(A) + 2\pi(A)\,\frac{h-\mathsf{P}_0 h_{out}}{1-2\mathsf{P}_0} & \text{if } u \in [\mathsf{P}_0, 1/2] \\ \pi(A) + h_{out}\,\pi(A) & \text{if } u \in [0, \mathsf{P}_0] \end{cases}$$

Bound λ_2. Rescale for general case.

$$\begin{aligned}\Longrightarrow \lambda_2 \;&\le\; -1 + \mathsf{P}_0\left(\sqrt{1-h_{in}} + \sqrt{1+h_{out}}\right) \\ &\quad + (1-\mathsf{P}_0)\left(\sqrt{1-\frac{h-\mathsf{P}_0 h_{in}}{1-\mathsf{P}_0}} + \sqrt{1+\frac{h-\mathsf{P}_0 h_{out}}{1-\mathsf{P}_0}}\right)\end{aligned}$$

Other Inequalities

Given h_{out}
Let $h = \mathsf{P}_0\, h_{out}$ and $h_{in} = h = \mathsf{P}_0\, h_{out}$.

$$\begin{aligned}
\implies \lambda_2 &\leq \sqrt{1 - \mathsf{P}_0\, h_{out}} + \mathsf{P}_0 \left(\sqrt{1 + h_{out}} - 1 \right) \\
&\leq 1 - \frac{\mathsf{P}_0\, h_{out}^2}{8} \left(1 + \mathsf{P}_0 - \frac{h_{out}}{2} \right)
\end{aligned}$$

Given $\tilde{h}_{in}$
Proceed as before, but consider $\tilde{h}_{in}(A)$ and then $\mathcal{C}_{\sqrt{x(1-x)}}$.

$$\begin{aligned}
\lambda_2 &\leq \sqrt{1 - \left(\frac{\tilde{h}_{in} \mathsf{P}_0}{2} \right)^2} + \mathsf{P}_0 \left(\sqrt{1 - \left(\frac{\tilde{h}_{in}}{2} \right)^2} - 1 \right) \\
&\leq 1 - \frac{\tilde{h}_{in}^2\, \mathsf{P}_0}{8} (1 + \mathsf{P}_0)
\end{aligned}$$

Consider $\mathcal{C}_{\sqrt{x(1-x)}}$ on previous page.

$$\begin{aligned}
\lambda_2 \leq\ & -1 + \mathsf{P}_0 \sqrt{(1 - h_{in}) \left(1 + h_{in} \frac{\pi_*}{1 - \pi_*} \right)} + \mathsf{P}_0 \sqrt{(1 + h_{out}) \left(1 - h_{out} \frac{\pi_*}{1 - \pi_*} \right)} \\
& + (1 - \mathsf{P}_0) \sqrt{\left(1 - \frac{h - \mathsf{P}_0 h_{in}}{1 - \mathsf{P}_0} \right) \left(1 + \frac{h - \mathsf{P}_0 h_{in}}{1 - \mathsf{P}_0} \frac{\pi_*}{1 - \pi_*} \right)} \\
& + (1 - \mathsf{P}_0) \sqrt{\left(1 + \frac{h - \mathsf{P}_0 h_{out}}{1 - \mathsf{P}_0} \right) \left(1 - \frac{h - \mathsf{P}_0 h_{out}}{1 - \mathsf{P}_0} \frac{\pi_*}{1 - \pi_*} \right)}
\end{aligned}$$

Periodic walk on uniform two-point space.

Proof of mixing theorem

Theorem (Montenegro) Every (non-lazy, non-reversible) Markov chain satisfies

$$\|\mathsf{P}^t(x,\cdot)-\pi\|_{TV} \le \frac{1}{\pi(x)}\,\mathsf{E}\pi(S_t)(1-\pi(S_t))$$

Proof: Let $\mathsf{K}^n(S_0,S_n)$ be probability of S_n given S_0.
Let $f(x)=(1-x)^+=(1-x)\,\delta_{x\le 1}$. Then

$$\begin{aligned}
\|\mathsf{p}^{(n)}-\pi\|_{TV} &= \sum_{y\in V} f\left(\frac{\mathsf{p}^{(n)}(y)}{\pi(y)}\right)\pi(y) && \text{set up problem}\\
&= \sum_{y\in V} f\left(\frac{\mathsf{P}_{\{x\}}(y\in S_n)}{\pi(x)}\right)\pi(y) && \text{introduce evolving sets}\\
&= \sum_{y\in V} f\left(\sum_{S_n\subset V}\frac{1_{y\in S_n}}{\pi(S_n)}\,\frac{\pi(S_n)\,\mathsf{K}^n(\{x\},S_n)}{\pi(x)}\right)\pi(y) && \text{rewrite with indicator}\\
&\le \sum_{y\in V}\sum_{S_n\subset V} f\left(\frac{1_{y\in S_n}}{\pi(S_n)}\right)\frac{\pi(S_n)\,\mathsf{K}^n(\{x\},S_n)}{\pi(x)}\,\pi(y) && \text{pull out sum with Jensen}\\
&= \sum_{S_n\subset V}\left[\sum_{y\in V} f\left(\frac{1_{y\in S_n}}{\pi(S_n)}\right)\pi(y)\right]\frac{\pi(S_n)\,\mathsf{K}^n(\{x\},S_n)}{\pi(x)} && \text{change order of summation}\\
&= \sum_{S_n\subset V}(1-\pi(S_n))\,\frac{\pi(S_n)\,\mathsf{K}^n(\{x\},S_n)}{\pi(x)} && \text{evaluate sum}
\end{aligned}$$

Remarks

Theorem (Zhang)

$$-\sqrt{1-\xi^2} \le \lambda_n \le -1 + 4\,\xi$$

where

$$\xi = \min_{S_1,\, S_2 \subset V} \frac{\mathsf{Q}(S_1,S_1) + \mathsf{Q}(S_2,S_2) + \mathsf{Q}(S_1 \cup S_2,\, V \setminus (S_1 \cup S_2))}{\pi(S_1 \cup S_2)}\,.$$

Conjecture If $|\lambda_n| > \lambda_2$ then $\phi < h$.

Conjecture The second smallest eigenvalue of the Laplacian of a graph is

$$\lambda_2 \;\ge\; 2\left[1 - \sqrt{1-(\tilde{i}^+_\infty/2)^2} + \Delta\left(1 - \sqrt{1-\left(\frac{\tilde{i}^+_\infty}{2\Delta}\right)^2}\right)\right]$$

$$\lambda_2 \;\ge\; 2\left[1 - \sqrt{1+i^-_\infty} + \Delta\left(1 - \sqrt{1-i^-_\infty/\Delta}\right)\right]$$